The Lyapunov stability of first order dynamic equations with respect to time-dependent Riemannian metrics

Gennadi Sardanashvily[1]

Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia

Abstract. Solutions of a smooth first order dynamic equation can be made Lyapunov stable at will by the choice of an appropriate time-dependent Riemannian metric.

1 Introduction

The notion of the Lyapunov stability of a dynamic equation on a smooth manifold implies that this manifold is equipped with a Riemannian metric. At the same time, no preferable Riemannian metric is associated to a first order dynamic equation. Here, we aim to study the Lyapunov stability of first order dynamic equations in non-autonomous mechanics with respect to different (time-dependent) Riemannian metrics.

Let us recall that a solution $s(t)$, $t \in \mathbb{R}$, of a first order dynamic equation is said to be Lyapunov stable (in the positive direction) if for $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there is $\delta > 0$ such that, if $s'(t)$ is another solution and the distance between the points $s(t_0)$ and $s'(t_0)$ is inferior to δ, then the distance between the points $s(t)$ and $s'(t)$ for all $t > t_0$ is inferior to ε. In order to formulate a criterion of the Lyapunov stability with respect to a time-dependent Riemannian metric, we introduce the notion of a covariant Lyapunov tensor as generalization of the well-known Lyapunov matrix. The latter is defined as the coefficient matrix of the variation equation [1, 3], and fails to be a tensor under coordinate transformations, unless they are linear and time-independent. On the contrary, the covariant Lyapunov tensor is a true tensor field, but it essentially depends on the choice of a Riemannian metric. We show the following (see Propositions 3, 5, and 6 below).

(i) If the covariant Lyapunov tensor is negative definite in a tubular neighbourhood of a solution s at points $t \geq t_0$, this solution is Lyapunov stable.

(ii) For any first order dynamic equation, there exists a (time-dependent) Riemannian metric such that every solution of this equation is Lyapunov stable.

(iii) Moreover, the Lyapunov exponent of any solution of a first order dynamic equation can be made equal to any real number with respect to the appropriate (time-dependent)

[1]E-mail address: sard@grav.phys.msu.su

Riemannian metric. It follows that chaos in dynamical systems described by smooth (C^∞) first order dynamic equations can be characterized in full by time-dependent Riemannian metrics.

2 Geometry of first order dynamic equations

Let $\mathbb{R}$ be the time axis provided with the Cartesian coordinate t and transition functions $t' = t$+const. In geometric terms [4], a (smooth) first order dynamic equation in non-autonomous mechanics is defined as a vector field γ on a smooth fibre bundle

$$\pi : Y \to \mathbb{R} \tag{1}$$

which obeys the condition $\gamma\rfloor dt = 1$, i.e.,

$$\gamma = \partial_t + \gamma^k \partial_k. \tag{2}$$

The associated first order dynamic equation takes the form

$$\dot{t} = 1, \qquad \dot{y}^k = \gamma^k(t, y^j)\partial_k, \tag{3}$$

where $(t, y^k, \dot{t}, \dot{y}^k)$ are holonomic coordinates on TY. Its solutions are trajectories of the vector field γ (2). They assemble into a (regular) foliation $\mathcal{F}$ of Y. Equivalently, γ (2) is defined as a connection on the fibre bundle (1).

A fibre bundle Y (1) is trivial, but it admits different trivializations

$$Y \cong \mathbb{R} \times M, \tag{4}$$

distinguished by fibrations $Y \to M$. For instance, if there is a trivialization (4) such that, with respect to the associated coordinates, the components γ^k of the connection γ (2) are independent of t, one says that γ is a conservative first order dynamic equation on M.

Hereafter, the vector field γ (2) is assumed to be complete, i.e., there is a unique global solution of the dynamic equation γ through each point of Y. For instance, if fibres of $Y \to \mathbb{R}$ are compact, any vector field γ (2) on Y is complete.

Proposition 1. *If the vector field γ (2) is complete, there exists a trivialization (4) of Y such that any solution s of γ reads*

$$s^a(t) = \text{const.}, \qquad t \in \mathbb{R},$$

with respect to associated bundle coordinates (t, y^a).

Proof. If γ is complete, the foliation $\mathcal{F}$ of its trajectories is a fibration ζ of Y along these trajectories onto any fibre of Y, e.g., $Y_{t=0} \cong M$. This fibration yields a desired trivialization [4]. □

One can think of the coordinates (t, y^a) in Proposition 1 as being the initial date coordinates because all points of the same trajectory differ from each other only in the temporal coordinate.

Let us consider the canonical lift $V\gamma$ of the vector field γ (2) onto the vertical tangent bundle VY of $Y \to \mathbb{R}$. With respect to the holonomic bundle coordinates $(t, y^k, \overline{y}^k)$ on VY, it reads

$$V\gamma = \gamma + \partial_j \gamma^k \overline{y}^j \overline{\partial}_k, \qquad \overline{\partial}_k = \frac{\partial}{\partial \overline{y}^k}.$$

This vector field obeys the condition $V\gamma \rfloor dt = 1$, and defines the first order dynamic equation

$$\dot{t} = 1, \qquad \dot{y}^k = \gamma^k(t, y^i), \tag{5a}$$

$$\dot{\overline{y}}_t{}^k = \partial_j \gamma^k(t, y^i) \overline{y}^j \tag{5b}$$

on VY. The equation (5a) coincides with the initial one (3). The equation (5b) is the well-known variation equation. Substituting a solution s of the initial dynamic equation (5a) into (5b), one obtains a linear dynamic equation whose solutions $\overline{s}$ are Jacobi fields of the solution s. In particular, if $Y \to \mathbb{R}$ is a vector bundle, there are the canonical splitting $VY \cong Y \times Y$ and the morphism $VY \to Y$ so that $s + \overline{s}$ obeys the initial dynamic equation (5a) modulo the terms of order > 1 in $\overline{s}$.

3 The covariant Lyapunov tensor

The collection of coefficients

$$l_j{}^k = \partial_j \gamma^k \tag{6}$$

of the variation equation (5b) is called the Lyapunov matrix. Clearly, it is not a tensor under bundle coordinate transformations of the fibre bundle Y (1). Therefore, we introduce a covariant Lyapunov tensor as follows.

Let a fibre bundle $Y \to \mathbb{R}$ be provided with a Riemannian fibre metric g, defined as a section of the symmetrized tensor product

$$\overset{2}{\vee} V^* Y \to Y \tag{7}$$

of the vertical cotangent bundle V^*Y of $Y \to \mathbb{R}$. With respect to the holonomic coordinates $(t, y^k, \overline{y}_k)$ on V^*Y, it takes the coordinate form

$$g = \frac{1}{2} g_{ij}(t, y^k) \overline{d}y^i \vee \overline{d}y^j,$$

where $\{\overline{d}y^i\}$ are the holonomic fibre bases for V^*Y.

Given a first order dynamic equation γ, let

$$V^*\gamma = \gamma - \partial_j \gamma^k \overline{y}_k \overline{\partial}^j, \qquad \overline{\partial}^j = \frac{\partial}{\partial \overline{y}_j}. \tag{8}$$

be the canonical lift of the vector field γ (2) onto V^*Y. It is a connection on $V^*Y \to \mathbb{R}$. Let us consider the Lie derivative $\mathbf{L}_\gamma g$ of the Riemannian fibre metric g along the vector field $V^*\gamma$ (8). It reads

$$L_{ij} = (\mathbf{L}_\gamma g)_{ij} = \partial_t g_{ij} + \gamma^k \partial_k g_{ij} + \partial_i \gamma^k g_{kj} + \partial_j \gamma^k g_{ik}. \tag{9}$$

This is a section of the fibre bundle (7) and, consequently, a tensor with respect to any bundle coordinate transformation of the fibre bundle (1). We agree to call it the covariant Lyapunov tensor. If g is an Euclidean metric, it comes to symmetrization

$$L_{ij} = \partial_i \gamma^j + \partial_j \gamma^i = l_i{}^j + l_j{}^i$$

of the Lyapunov matrix (6).

Let us point the following two properties of the covariant Lyapunov tensor.

(i) Written with respect to the initial date coordinates in Proposition 1, the covariant Lyapunov tensor is

$$L_{ab} = \partial_t g_{ab}.$$

(ii) Given a solution s of the dynamic equation γ and a solution $\overline{s}$ of the variation equation (5b), we have

$$L_{ij}(t, s^k(t)) \overline{s}^i \overline{s}^j = \frac{d}{dt}(g_{ij}(t, s^k(t)) \overline{s}^i \overline{s}^j).$$

The definition of the covariant Lyapunov tensor (9) depends on the choice of a Riemannian fibre metric on the fibre bundle Y.

Proposition 2. *If the vector field γ is complete, there is a Riemannian fibre metric on Y such that the covariant Lyapunov tensor vanishes everywhere.*

Proof. Let us choose the atlas of the initial date coordinates in Proposition 1. Using the fibration $\zeta : Y \to Y_{t=0}$, one can provide Y with a time-independent Riemannian fibre metric

$$g_{ab}(t, y^c) = h(t) g_{ab}(0, y^c) \tag{10}$$

where $g_{ab}(0, y^c)$ is a Riemannian metric on the fibre $Y_{t=0}$ and $h(t)$ is a positive smooth function on $\mathbb{R}$. The covariant Lyapunov tensor with respect to the metric (10) is

$$L_{ab} = \partial_t h g_{ab}.$$

Putting $h(t) = 1$, we obtain $L = 0$. □

4 The Lyapunov stability of a first order dynamic equation

With the covariant Lyapunov tensor (9), we obtain the following criterion of the stability condition of Lyapunov.

Recall that, given a Riemannian fibre metric g on a fibre bundle $Y \to \mathbb{R}$, the instantwise distance $\rho_t(s, s')$ between two solutions s and s' of a dynamic equation γ on Y at an instant t is the distance between the points $s(t)$ and $s'(t)$ in the Riemannian space $(Y_t, g(t))$.

Proposition 3. *Let s be a solution of a first order dynamic equation γ. If there exists an open tubular neighbourhood U of the trajectory s where the covariant Lyapunov tensor (9) is negative-definite at all instants $t \geq t_0$, then there exists an open tubular neighbourhood U' of s such that*

$$\lim_{t' \to \infty} [\rho_{t'}(s, s') - \rho_t(s, s')] < 0$$

for any $t > t_0$ and any solution s' crossing U'.

Proof. Since the condition and the statement of Proposition are coordinate-independent, let us choose the following chart of initial date coordinates in Proposition 1 which cover the trajectory s. Put $t = 0$ without a loss of generality. There is an open neighbourhood $U_0 \subset Y_0 \cap U$ of $s(0)$ in the Riemannian manifold $(Y_0, g(0))$ which can be provided with the normal coordinates (x^a) defined by the Riemannian metric $g(0)$ in Y_0 and centralized at $s(0)$. Let us consider the open tubular $U' = \zeta^{-1}(U_0)$ endowed with the coordinates (t, x^a). It is the desired chart of initial date coordinates. With respect to these coordinates, the solution s reads $s^a(t) = 0$. Let

$$s'^a(t) = u^a = \text{const}.$$

be another solution crossing U'. The instantwise distance $\rho_t(s,s')$, $t\geq 0$, between solutions s and s' is the distance between the points $(t,0)$ and (t,u) in the Riemannian space $(Y_t, g(t))$. This distance does not exceed the length

$$\overline{\rho}_t(s,s') = \left[\int\limits_0^1 g_{ab}(t,\tau u^c)u^a u^b d\tau\right]^{1/2} \tag{11}$$

of the curve

$$x^a = \tau u^a, \qquad \tau\in[0,1] \tag{12}$$

in the Riemannian space $(Y_t, g(t))$, while

$$\rho_0(s,s') = \overline{\rho}_0(s,s')$$

The temporal derivative of the function $\overline{\rho}_t(s,s')$ (11) reads

$$\partial_t\overline{\rho}_t(s,s') = \frac{1}{2(\overline{\rho}_t(s,s'))^{1/2}}\int\limits_0^1 \partial_t g_{ab}(t,\tau u^c)u^a u^b d\tau. \tag{13}$$

Since the bilinear form $\partial_t g_{ab} = L_{ab}$, $t\geq 0$, is negative-definite at all points of the curve (12), the derivative (13) at all points $t\geq t_0$ is also negative. Hence, we obtain

$$\rho_{t>0}(s,s') < \overline{\rho}_{t>0}(s,s') < \overline{\rho}_0(s,s') = \rho_0(s,s').$$

□

Corollary 4. *The solution s obeying the condition of Proposition 3 is Lyapunov stable with respect to the Riemannian fibre metric g.*

It is readily observed that Proposition 3 states something more. One can think of the solution s in Proposition 3 as being isometrically Lyapunov stable. Of course, being Lyapunov stable with respect a Riemannian fibre metric g, a solution s need not be so with respect to another Riemannian fibre metric g', unless g' results from g by a time-independent transformation.

Proposition 5. *For any first order dynamic equation defined by a complete vector field γ (2) on a fibre bundle $Y\to\mathbb{R}$, there exists a Riemannian fibre metric on Y such that each solution of γ is Lyapunov stable.*

Proof. This property obviously holds with respect to the Riemannian fibre metric (10) in Proposition 2 where $h=1$. □

Proposition 5 can be improved as follows.

Proposition 6. *Let λ be a real number. Given a dynamic equation γ defined by a complete vector field γ (2), there is a Riemannian fibre metric on Y such that the Lyapunov spectrum of any solution of γ reduces to λ.*

Proof. Recall that the (upper) Lyapunov exponent of a solution s' with respect to a solution s is defined as the limit

$$K(s,s') = \overline{\lim_{t\to\infty}} \frac{1}{t}\ln(\rho_t(s,s')). \tag{14}$$

Let us provide Y with the Riemannian fibre metric (10) in Proposition 2 where $h = \exp(\lambda t)$. A simple computation shows that the Laypunov exponent (14) with respect to this metric is exactly λ. □

If the upper limit

$$\overline{\lim_{\rho_{t=0}(s,s')\to 0}} K(s,s') = \lambda$$

is negative, the solution s is said to be exponentially Lyapunov stable. If there exists at least one positive Lyapunov exponent, one speaks about chaos in a dynamical system [2]. Proposition 6 shows that chaos in smooth dynamical systems can be characterized in full by time-dependent Riemannian metrics.

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